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Quantization of generalized Virasoro-like algebras[☆]

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Abstract

In a recent paper by the authors, Lie bialgebras structures of generalized Virasoro-like type were considered. In this paper, the explicit formula of the quantization of generalized Virasoro-like algebras is presented. © 2008 Elsevier Inc. All rights reserved.

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1. Introduction

In Hopf algebras or quantum groups theory, there are two standard methods to yield new bialgebras from old ones, one is by twisting the product by a 2-cocycle but keeping the coproduct unchanged, another is by twisting the coproduct by a Drinfel'd twist element but keeping the product unchanged. It is known that the construction of quantizations of Lie bialgebras is an important method to produce new quantum groups (cf. [1,5], etc). The quantizations of Lie algebras have close relations with their bialgebra structures. In the papers [2,3] the author proved that there is a 1–1 correspondence between the unitary solutions in $g \otimes g$ of the classical Yang-Baxter equation (where g is a Lie algebra) and the equivalence classes of Drinfel'd elements

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based on $\mathcal{U}(g)[[t]]$ (if \mathcal{F} is a Drinfel'd element and $f = 1 + tf_1 + \cdots + t^n f_n + \cdots \in \mathcal{U}(g)[[t]]$ then $\mathcal{F}' = (\Delta(f))(\mathcal{F})(f^{-1} \otimes f^{-1})$ is also a Drinfel'd element; in this case \mathcal{F} and \mathcal{F}' are called *equivalent*). However, it is difficult to compute the Drinfel'd element from the unitary solutions in $g \otimes g$ of the classical Yang-Baxter equation. This is probably the reason why we know so few constructions of quantizations of Lie bialgebras. In the paper [10] (cf. [11,12,17]), a class of infinite dimensional Lie bialgebras containing Virasoro algebras were presented. This type Lie bialgebras were classified in [14], and the quantizations of this type Lie algebras were determined in [7]. In the paper [15], the structures of Lie bialgebras of generalized Witt type were considered. The quantizations of this type Lie bialgebras were investigated in [8].

Recently the Virasoro-like algebra has been a subject of intensive study (see, e.g., [9,13,22] and references therein). This is probably because the Virasoro-like algebra is not only closely related to the Virasoro algebra, but also it is a special case of Lie algebras of Block type (cf. [4,20]) and a special case of Cartan type S (cf. [16,19,21]), meanwhile it can also be regarded as a special case of its q -analog when q tends to 1 (cf. [22]). In the previous paper [18], we considered the structures of Lie bialgebras of generalized Virasoro-like algebras. In the present paper, we shall consider the quantizations of this type of Lie bialgebras over an arbitrary field of characteristic zero. From Theorem 2.2 we shall see that all of the quantized constructions related to the Virasoro-like bialgebra which are quantized by Drinfel'd elements can be determined by Theorem 2.6.

2. Preliminaries

2.1. Generalized Virasoro-like Lie bialgebras

Throughout this paper, \mathbb{F} denotes an arbitrary field of characteristic zero. Let Γ be any nondegenerate additive subgroup of \mathbb{F}^2 (namely Γ contains an \mathbb{F} -basis of \mathbb{F}^2).

Definition 2.1. The Lie algebra $\mathcal{L}(\Gamma)$ with basis $\{L_\alpha, \partial_1, \partial_2 | \alpha = (\alpha_1, \alpha_2) \in \Gamma \setminus \{0\}\}$ and bracket:

$$[L_\alpha, L_\beta] = (\alpha_1\beta_2 - \alpha_2\beta_1)L_{\alpha+\beta}, \quad [\partial_i, L_\alpha] = \alpha_i L_\alpha \quad \text{for } \alpha, \beta \in \Gamma \setminus \{0\}, i = 1, 2,$$

is called a *generalized Virasoro-like algebra*. In particular, when $\Gamma = \mathbb{Z}^2$, the derived subalgebras $[\mathcal{L}(\mathbb{Z}^2), \mathcal{L}(\mathbb{Z}^2)] = \text{span}\{L_\alpha | \alpha \in \mathbb{Z}^2 \setminus \{0\}\}$ is the Virasoro-like algebra (cf. [9,13]).

Below we shall use the convention that if an undefined notation appears in an expression, we always treat it as zero; for instance, $L_\alpha = 0$ if $\alpha = 0$.

The following theorem is the main result in [18].

Theorem 2.2. *Every Lie bialgebra structure on the Lie algebras $\mathcal{L}(\Gamma)$ is a coboundary triangular Lie bialgebra.*

2.2. Drinfel'd twisting

Let A be a unital R -algebra (where R is a ring). For any element $x \in A, a \in R$, we set

$$x_a^{(n)} = (x+a)(x+a+1) \cdots (x+a+n-1), \quad (2.1)$$

$$x_a^{[n]} = (x+a)(x+a-1) \cdots (x+a-n+1), \quad (2.2)$$

where $n \in \mathbb{Z}$, $x_a^{[0]} = x_a^{<0>} = 1$, and $x_a^{[n]} = x_a^{<n>} = 0$ if $n < 0$. We also denote $x^{<n>} = x_0^{<n>}$, $x^{[n]} = x_0^{[n]}$.

The following lemma belongs to [7,6].

Lemma 2.3. *Let x be any element of a unital \mathbb{F} -algebras A . For $a, d \in \mathbb{F}$ and $m, n, r \in \mathbb{Z}_+$, one has*

$$x_a^{<m+n>} = x_a^{<m>} x_{m+a}^{<n>}, \quad (2.3)$$

$$x_a^{[m+n]} = x_a^{[m]} x_{a-m}^{[n]}, \quad (2.4)$$

$$x_a^{[m]} = x_{a-m+1}^{<m>}, \quad (2.5)$$

$$\sum_{m+n=r} \frac{(-1)^n}{m!n!} x_a^{[m]} x_d^{<n>} = \binom{a-d}{r} = \frac{(a-d)(a-d+1) \cdots (a-d-r+1)}{r!}, \quad (2.6)$$

$$\begin{aligned} \sum_{m+n=r} \frac{(-1)^m}{m!n!} x_a^{[m]} x_{d-m}^{[n]} &= \binom{a-d+r-1}{r} \\ &= \frac{(a-d)(a-d+1) \cdots (a-d+r-1)}{r!}. \end{aligned} \quad (2.7)$$

The following definition belongs to [2,3].

Definition 2.4. Let $(\mathcal{H}, \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be a Hopf algebra over a commutative ring. An element $\mathcal{F} \in H \otimes H$ is called a Drinfel'd twisting element of \mathcal{H} if it is invertible such that

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes Id)(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)(\mathcal{F}), \quad (2.8)$$

$$(\epsilon_0 \otimes Id)(\mathcal{F}) = 1 \otimes 1 = (Id \otimes \epsilon_0)(\mathcal{F}). \quad (2.9)$$

The following well-known theorem can be found in [2,3,5].

Theorem 2.5. *Let $(\mathcal{H}, \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be a Hopf algebra over commutative ring, \mathcal{F} a Drinfel'd element of \mathcal{H} . Then*

- (1) $u = \mu(Id \otimes S_0)(\mathcal{F})$ is an invertible element of $\mathcal{H} \otimes \mathcal{H}$ with $u^{-1} = \mu(S_0 \otimes Id)(\mathcal{F})$.
- (2) The algebras $(\mathcal{H}, \mu, \tau, \Delta, \epsilon, S)$ is a new Hopf algebra if we keep the counit undeformed (i.e., $\epsilon = \epsilon_0$) and define $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$, $S : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\Delta(h) = \mathcal{F} \Delta_0(h) \mathcal{F}^{-1}, \quad S(h) = u S_0(h) u^{-1}.$$

Let $(\mathcal{U}(\mathcal{L}(\Gamma)), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be the standard Hopf algebra, i.e.

$$\begin{aligned} \Delta_0(L_\alpha) &= L_\alpha \otimes 1 + 1 \otimes L_\alpha, \quad \Delta_0(\partial_j) = \partial_j \otimes 1 + 1 \otimes \partial_j, \\ S_0(L_\alpha) &= -L_\alpha, \quad S_0(\partial_j) = -\partial_j, \quad \epsilon_0(L_\alpha) = 0, \quad \epsilon_0(\partial_j) = 0, \end{aligned}$$

for $\alpha \in \Gamma \setminus \{0\}$, $j = 1, 2$. The main result of this paper is the following theorem.

Theorem 2.6. *Let $\mathcal{L}(\Gamma)$ be the generalized Virasoro-like algebra. For $\alpha \in \Gamma \setminus \{0\}$, $L_\alpha \in \mathcal{L}(\Gamma)$, choose $T = a_1 \partial_1 + a_2 \partial_2 \in \text{span}\{\partial_1, \partial_2\}$ such that $[T, L_\alpha] = L_\alpha$. Then there exists a noncommu-*

tative and noncocommutative Hopf algebra structure $(\mathcal{U}(\mathcal{L}(\Gamma))[[t]], \mu, \tau, \Delta, S, \epsilon)$ on $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]$ over $\mathbb{F}[[t]]$ such that $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]/t\mathcal{U}(\mathcal{L}(\Gamma))[[t]] \cong \mathcal{U}(\mathcal{L}(\Gamma))$, which preserves the product and the counit of $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]$, but the coproduct and antipode are defined by

$$\Delta(L_\beta) = L_\beta \otimes (1 - L_\alpha t)^b + \sum_{i=0}^{\infty} (-1)^i T^{(i)} \otimes (1 - L_\alpha t)^{-i} L_{\beta+i\alpha} c_i t^i, \quad (2.10)$$

$$\Delta(\partial_j) = \partial_j \otimes 1 + 1 \otimes \partial_j + \alpha_j T \otimes (1 - L_\alpha t)^{-1} L_\alpha t, \quad j = 1, 2, \quad (2.11)$$

$$S(L_\beta) = -(1 - L_\alpha t)^{-b} \sum_{i=0}^{\infty} L_{\beta+i\alpha} c_i T_1^{(i)} t^i, \quad (2.12)$$

$$S(\partial_j) = \alpha_j T (1 - L_\alpha t)^{-1} (L_\alpha t - L_\alpha^2 t^2) - \partial_j, \quad (2.13)$$

where for any $\beta = (\beta_1, \beta_2) \in \Gamma \setminus \{0\}$, we denote $b = a_1 \beta_1 + a_2 \beta_2$, $c_i = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)^i}{i!}$, $c_0 = 1$, $j = 1, 2$.

3. Proof of the main result

We shall divide the proof of Theorem 2.6 into several lemmas.

Lemma 3.1. For any $L_\alpha \in \mathcal{L}(\Gamma)$ with $\alpha \neq 0$, we choose $T = a_1 \partial_1 + a_2 \partial_2$ with $a_1, a_2 \in \mathbb{F}$ such that

$$[T, L_\alpha] = [a_1 \partial_1 + a_2 \partial_2, L_\alpha] = L_\alpha.$$

For any $\beta = (\beta_1, \beta_2) \in \Gamma \setminus \{0\}$, we denote $b = a_1 \beta_1 + a_2 \beta_2$. Then the following equations hold in $\mathcal{U}(\mathcal{L}(\Gamma))$ for $a \in \mathbb{F}$, $m, k \in \mathbb{Z}_+$,

$$L_\beta T_a^{[m]} = T_{a-b}^{[m]} L_\beta, \quad (3.1)$$

$$L_\beta T_a^{(m)} = T_{a-b}^{(m)} L_\beta, \quad (3.2)$$

$$L_\alpha^k T_a^{[m]} = T_{a-k}^{[m]} L_\alpha^k, \quad (3.3)$$

$$L_\alpha^k T_a^{(m)} = T_{a-k}^{(m)} L_\alpha^k, \quad (3.4)$$

$$\partial_j^k T_a^{[m]} = T_a^{[m]} \partial_j^k, \quad j = 1, 2, \quad (3.5)$$

$$\partial_j^k T_a^{(m)} = T_a^{(m)} \partial_j^k, \quad j = 1, 2, \quad (3.6)$$

$$L_\beta L_\gamma^m = \sum_{i=0}^m (-1)^i \binom{m}{i} (\gamma_1 \beta_2 - \gamma_2 \beta_1)^i L_\gamma^{m-i} L_{\beta+i\gamma}, \quad (3.7)$$

$$\partial_j L_\gamma^m = m \gamma_j L_\gamma^{m-1} + L_\gamma^m \partial_j, \quad j = 1, 2. \quad (3.8)$$

Proof. Since

$$[T, L_\beta] = [a_1 \partial_1 + a_2 \partial_2, L_\beta] = (a_1 \beta_1 + a_2 \beta_2) L_\beta = b L_\beta = T L_\beta - L_\beta T,$$

we have $L_\beta T = (T - b) L_\beta$. It easy to see that (3.1) is true for $m = 1$. Suppose that (3.1) is true for m . Then for $m + 1$, we have

$$\begin{aligned} L_\beta T_a^{[m+1]} &= L_\beta T_a^{[m]} (T + a - m) = T_{a-b}^{[m]} L_\beta (T + a - m) \\ &= T_{a-b}^{[m]} (T + a - b - m) L_\beta = T_{a-b}^{[m+1]}. \end{aligned}$$

By induction on m , (3.1) holds. Similarly, we can obtain (3.2)–(3.4). Since $[\partial_j, T] = 0$, the equations (3.5), (3.6) are obviously. For (3.7), we have

$$\begin{aligned} L_\beta L_\gamma^m &= \sum_{i=0}^m (-1)^i \binom{m}{i} L_\gamma^{m-i} (\text{ad} L_\gamma)^i (L_\beta) \\ &= \sum_{i=0}^m (-1)^i \binom{m}{i} L_\gamma^{m-i} L_{\beta+i\gamma} (\gamma_1 \beta_2 - \gamma_2 \beta_1)^i. \end{aligned}$$

The proof of formula (3.8) is similar to that of (3.7), using the following fact:

$$(\text{ad} L_\gamma)^i (\partial_j) = \begin{cases} -\gamma_j L_\gamma, & \text{if } i = 1, \\ 0, & \text{if } i > 1. \end{cases} \quad \square$$

We remark that in case $T = a_1 \partial_1 + a_2 \partial_2 \in \text{span}\{\partial_1, \partial_2\}$ in the above lemma such that $[T, L_\alpha] = (a_1 \alpha_1 + a_2 \alpha_2) L_\alpha$, for $\alpha = (\alpha_1, \alpha_2) \in \Gamma$ with $c := \alpha_1 \alpha_1 + a_2 \alpha_2 \neq 0, 1$, then we can replace T by $c^{-1} T$.

Now for any $a \in \mathbb{F}$, we denote

$$\begin{aligned} \mathcal{F}_a &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes L_\alpha^i t^i, \\ F_a &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_\alpha^i t^i, \\ u_a &= \mu \cdot (S_0 \otimes Id)(F_a), \quad v_a = \mu \cdot (Id \otimes S_0)(\mathcal{F}_a). \end{aligned}$$

Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$. Since $S_0(T_a^{(i)}) = (-1)^i T_{-a}^{[i]}$, $S_0(L_\alpha^i) = (-1)^i L_\alpha^i$, we have

$$\begin{aligned} u_a &= \mu(S_0 \otimes Id) \left(\sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_\alpha^i t^i \right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} L_\alpha^i t^i, \\ v_a &= \mu(Id \otimes S_0) \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes L_\alpha^i t^i \right) = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} L_\alpha^i t^i. \end{aligned}$$

Lemma 3.2. For $a, d \in \mathbb{F}$, one has

$$\mathcal{F}_a F_d = 1 \otimes (1 - L_\alpha t)^{(a-d)}, \quad v_a u_d = (1 - L_\alpha t)^{-(a+d)}.$$

Therefore the elements $\mathcal{F}_a, F_a, u_a, v_a$ are invertible with $\mathcal{F}_a^{-1} = F_a, u_a^{-1} = v_{-a}$.

Proof. Using the formula (2.6), we have

$$\begin{aligned} \mathcal{F}_a F_d &= \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes L_\alpha^i t^i \right) \cdot \left(\sum_{j=0}^{\infty} \frac{1}{j!} T_d^{(j)} \otimes L_\alpha^j t^j \right) \\ &= \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i! j!} T_a^{[i]} T_d^{(j)} \otimes L_\alpha^i L_\alpha^j t^{i+j} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_d^{[j]} \right) \otimes L_{\alpha}^m t^m \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{a-d}{m} \otimes L_{\alpha}^m t^m = 1 \otimes (1 - L_{\alpha} t)^{a-d}, \\
v_a u_d &= \left(\sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} L_{\alpha}^i t^i \right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} T_{-d}^{[j]} L_{\alpha}^j t^j = \sum_{i,j=0}^{\infty} \frac{1}{i!} T_a^{[i]} L_{\alpha}^i t^i \frac{(-1)^j}{j!} T_{-d}^{[j]} L_{\alpha}^j t^j \\
&= \sum_{m=0}^{\infty} \sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_{-d}^{[j]} L_{\alpha}^{i+j} t^{i+j} \quad (\text{from (3.3)}) \\
&= \sum_{m=0}^{\infty} \binom{a+d+m-1}{m} L_{\alpha}^m t^m \quad (\text{from (2.7)}) \\
&= (1 - L_{\alpha} t)^{-(a+d)}. \quad \square
\end{aligned}$$

Lemma 3.3. For any nonnegative integer m and any $a \in \mathbb{F}$, we have

$$\Delta_0 T^{[m]} = \sum_{i=0}^m \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]}.$$

In particular, we have $\Delta_0 T^{[m]} = \sum_{i=0}^m \binom{m}{i} T^{[i]} \otimes T^{[m-i]}.$

Proof. Since $\Delta_0(T) = T \otimes 1 + 1 \otimes T$, it is easy to see that the result is true for $m = 1$. Suppose it is true for m . Then for $m + 1$, we have

$$\begin{aligned}
\Delta_0(T^{[m+1]}) &= \Delta_0(T^{[m]}) \Delta_0(T - m) \\
&= \left(\sum_{i=0}^m \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \right) \\
&\quad \times ((T - a - m) \otimes 1 + 1 \otimes (T + a - m) + m(1 \otimes 1)) \\
&= \left(\sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \right) ((T - a - m) \otimes 1 + 1 \otimes (T + a - m)) \\
&\quad + m \left(\sum_{i=0}^m \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \right) + (1 \otimes T_a^{[m+1]} + T_{-a}^{[m+1]} \otimes 1) \\
&\quad + (T - a - m) \otimes T_a^{[m]} + T_{-a}^{[m]} \otimes (T + a - m) \\
&= 1 \otimes T_a^{[m+1]} + T_{-a}^{[m+1]} \otimes 1 + m \left(\sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \right) \\
&\quad + (T - a) \otimes T_a^{[m]} + T_{-a}^{[m]} \otimes (T + a) + \sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i+1]} \otimes T_a^{[m-i]}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m-1} (i-m) \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} + \sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i+1]} \\
& + \sum_{i=1}^{m-1} (-i) \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \\
& = 1 \otimes T_a^{[m+1]} + T_{-a}^{[m+1]} \otimes 1 + \sum_{i=1}^m \left(\binom{m}{i-1} + \binom{m}{i} \right) T_{-a}^{[i]} \otimes T_a^{[m+1-i]} \\
& = \sum_{i=0}^{m+1} \binom{m+1}{i} T_{-a}^{[i]} \otimes T_a^{[m+1-i]}.
\end{aligned}$$

By induction on m , the result holds. \square

Let $(\mathcal{U}(\mathcal{L}(\Gamma)), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be the standard Hopf algebra. Then $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]$ has a natural Hopf algebra structure which is induced from $(\mathcal{U}(\mathcal{L}(\Gamma)), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ by t -linear expansion, and we denote it by $(\mathcal{U}(\mathcal{L}(\Gamma))[[t]], \mu, \tau, \Delta_0, \epsilon_0, S_0)$. Then

Lemma 3.4. $\mathcal{F} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T^{[i]} \otimes L_{\alpha}^i t^i$ is a Drinfel'd twist element of $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]$, i.e.

$$\begin{aligned}
(\mathcal{F} \otimes 1)(\Delta_0 \otimes Id)(\mathcal{F}) &= (1 \otimes \mathcal{F})(1 \otimes \Delta_0)(\mathcal{F}), \\
(\epsilon_0 \otimes Id)(\mathcal{F}) &= 1 \otimes 1 = (Id \otimes \epsilon_0)(\mathcal{F}).
\end{aligned}$$

Proof. The second equation holds obviously, thus we just need to prove the first one. Since

$$\begin{aligned}
(\mathcal{F} \otimes 1)(\Delta_0 \otimes Id)(\mathcal{F}) &= \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T^{[i]} \otimes L_{\alpha}^i t^i \otimes 1 \right) \\
&\quad \times (\Delta_0 \otimes Id) \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} T^{[j]} \otimes L_{\alpha}^j t^j \right) \\
&= \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T^{[i]} \otimes L_{\alpha}^i t^i \otimes 1 \right) \\
&\quad \times \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^j \binom{j}{k} T_{-i}^{[k]} \otimes T_i^{[j-k]} \otimes L_{\alpha}^j t^j \right) \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!j!} t^{i+j} \sum_{k=0}^j \binom{j}{k} T^{[i]} T_{-i}^{[k]} \otimes L_{\alpha}^i T_i^{[j-k]} \otimes L_{\alpha}^j \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!j!} t^{i+j} \sum_{k=0}^j \binom{j}{k} T^{[i+k]} \otimes T^{[j-k]} L_{\alpha}^i \otimes L_{\alpha}^j,
\end{aligned}$$

on the other hand,

$$\begin{aligned}
 (1 \otimes \mathcal{F})(Id \otimes \Delta_0)(\mathcal{F}) &= \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} t^r 1 \otimes T^{[r]} \otimes L_{\alpha}^r \right) \\
 &\quad \times \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} t^s T^{[s]} \otimes \sum_{q=0}^s \binom{s}{q} L_{\alpha}^q \otimes L_{\alpha}^{s-q} \right) \\
 &= \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}}{r!s!} t^{r+s} \sum_{q=0}^s \binom{s}{q} T^{[s]} \otimes T^{[r]} L_{\alpha}^q \otimes L_{\alpha}^{r+s-q},
 \end{aligned}$$

it sufficient to show for a fixed m that

$$\begin{aligned}
 &\sum_{i+j=m} \frac{1}{i!j!} t^{i+j} \sum_{k=0}^j \binom{j}{k} T^{[i+k]} \otimes T^{[j-k]} L_{\alpha}^i \otimes L_{\alpha}^j \\
 &= \sum_{r+s=m} \frac{1}{r!s!} t^{r+s} \sum_{q=0}^s \binom{s}{q} T^{[s]} \otimes T^{[r]} L_{\alpha}^q \otimes L_{\alpha}^{r+s-q}.
 \end{aligned}$$

Now fix $r, 0 \leq i \leq s$, and set $i = q, i + k = s$. Then we have

$$j - k = j - (s - i) = i + j - s = m - s = r.$$

We see that the coefficients of $T^{[s]} \otimes T^{[r]} L_{\alpha}^q \otimes L_{\alpha}^{m-q}$ in both sides are equal. So the result holds. \square

Lemma 3.5. For $a \in \mathbb{F}, \beta \in \Gamma \setminus \{0\}$, we have

$$(L_{\beta} \otimes 1)F_a = F_{a-b}(L_{\beta} \otimes 1), \quad (3.9)$$

$$(1 \otimes L_{\beta})F_a = \sum_{l=0}^{\infty} (-1)^l F_{a+l} (T_a^{(l)} \otimes c_l L_{\beta+l\alpha} t^l), \quad (3.10)$$

$$L_{\beta} u_a = u_{a+b} \sum_{l=0}^{\infty} L_{\beta+l\alpha} c_l T_{1-a}^{(l)} t^l, \quad (3.11)$$

$$(\partial_j \otimes 1)F_a = F_a(\partial_j \otimes 1), \quad (3.12)$$

$$(1 \otimes \partial_j)F_a = F_{a+1}(T_a^{(1)} \otimes \alpha_j L_{\alpha} t) + F_a(1 \otimes \partial_j), \quad (3.13)$$

$$\partial_j u_a = -\alpha_j T_{-a}^{[1]} u_{a+1} L_{\alpha} t + u_a \partial_j, \quad (3.14)$$

$$L_{\alpha} u_a = u_{a+1} L_{\alpha}, \quad (3.15)$$

$$v T_{-a}^{[1]} = T_{-a}^{[1]} v_a - T_a^{[1]} v_{a-1} L_{\alpha} t, \quad (3.16)$$

where $j = 1, 2$, and $c_l = \frac{1}{l!}(\alpha_1 \beta_2 - \alpha_2 \beta_1)^l, b = a_1 \beta_1 + a_2 \beta_2$ for $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$.

Proof. From (3.2), we have

$$\begin{aligned}
 (L_\beta \otimes 1)F_a &= (L_\beta \otimes 1) \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_\alpha^i t^i \\
 &= \sum_{i=0}^{\infty} \frac{1}{i!} L_\beta T_a^{(i)} \otimes L_\alpha^i t^i \\
 &= \sum_{i=0}^{\infty} \frac{1}{i!} T_{a-b}^{(i)} L_\beta \otimes L_\alpha^i t^i = F_{a-b}(L_\beta \otimes 1).
 \end{aligned}$$

This proves (3.9). For (3.10), using (3.7), we have

$$\begin{aligned}
 (1 \otimes L_\beta)F_a &= (1 \otimes L_\beta) \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_\alpha^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_\beta L_\alpha^i t^i \\
 &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes \sum_{l=0}^i (-1)^l \binom{i}{l} L_\alpha^{i-l} L_{\beta+l\alpha} (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l t^i \\
 &= \sum_{i=0}^{\infty} \left(\sum_{l=0}^i (-1)^l \frac{1}{(i-l)! l!} T_a^{(i)} \otimes L_\alpha^{i-l} L_{\beta+l\alpha} (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l \right) t^i \\
 &= \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{1}{i! l!} T_a^{(i+l)} \otimes L_\alpha^i L_{\beta+l\alpha} (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l t^{i+l} \\
 &= \sum_{l=0}^{\infty} (-1)^l \sum_{i=0}^{\infty} \left(\frac{1}{i!} T_{a+l}^{(i)} \otimes L_\alpha^i t^i \right) \frac{1}{l!} T_a^{(l)} \otimes L_{\beta+l\alpha} t^l (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l \\
 &= \sum_{l=0}^{\infty} (-1)^l F_{a+l} (T_a^{(l)} \otimes L_{\beta+l\alpha} c_l t^l).
 \end{aligned}$$

So we have (3.10). And since

$$\begin{aligned}
 L_\beta u_a &= L_\beta \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} L_\alpha^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} L_\beta T_{-a}^{[r]} L_\alpha^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-b}^{[r]} L_\beta L_\alpha^r t^r \\
 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-b}^{[r]} \sum_{l=0}^r (-1)^l \binom{r}{l} L_\alpha^{r-l} (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l L_{\beta+l\alpha} t^r \\
 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-b}^{[r]} \sum_{l=0}^r (-1)^l \frac{r!}{(r-l)! l!} L_\alpha^{r-l} (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l L_{\beta+l\alpha} t^r \\
 &= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r! l!} T_{-a-b}^{[r+l]} L_\alpha^r (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l L_{\beta+l\alpha} t^{r+l} \\
 &= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r! l!} T_{-a-b}^{[r]} T_{-a-b-r}^{[l]} L_\alpha^r (\alpha_1 \beta_2 - \alpha_2 \beta_1)^l L_{\beta+l\alpha} t^{r+l}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \left(\frac{(-1)^r}{r!} T_{-a-b}^{[r]} L_{\alpha}^r t^r \right) T_{-a-b}^{[l]} L_{\beta+l\alpha} \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)^l}{l!} t^l \\
&= u_{a+b} \sum_{l=0}^{\infty} T_{-a-b}^{[l]} L_{\beta+l\alpha} c_l t^l = u_{a+b} \sum_{l=0}^{\infty} L_{\beta+l\alpha} T_{-a+l}^{[l]} c_l t^l \\
&= u_{a+b} \sum_{l=0}^{\infty} L_{\beta+l\alpha} T_{1-a}^{(l)} c_l t^l,
\end{aligned}$$

the formula (3.11) is obtained. For (3.12), we have

$$\begin{aligned}
(\partial_j \otimes 1)F_a &= (\partial_j \otimes 1) \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_{\alpha}^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} \partial_j T_a^{(i)} \otimes L_{\alpha}^i t^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \partial_j \otimes L_{\alpha}^i t^i = \left(\sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_{\alpha}^i t^i \right) (\partial_j \otimes 1) = F_a (\partial_j \otimes 1).
\end{aligned}$$

Using (2.3) and (3.8), we have

$$\begin{aligned}
(1 \otimes \partial_j)F_a &= (1 \otimes \partial_j) \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_{\alpha}^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes \partial_j L_{\alpha}^i t^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes (i\alpha_j L_{\alpha}^i + L_{\alpha}^i \partial_j) t^i \\
&= \sum_{i=1}^{\infty} \frac{1}{(i-1)!} T_a^{(i)} \otimes \alpha_j L_{\alpha}^i t^i + \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{(i)} \otimes L_{\alpha}^i \partial_j t^i \\
&= \sum_{i=1}^{\infty} \frac{1}{(i-1)!} T_a^{(1)} T_{a+1}^{(i-1)} \otimes \alpha_j L_{\alpha}^i t^i + F_a (1 \otimes \partial_j) \\
&= F_{a+1} (T_a^{(1)} \otimes \alpha_j L_{\alpha} t) + F_a (1 \otimes \partial_j).
\end{aligned}$$

So (3.13) holds. For (3.14), we have

$$\begin{aligned}
\partial_j u_a &= \partial_j \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} L_{\alpha}^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} \partial_j L_{\alpha}^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} (r\alpha_j L_{\alpha}^r + L_{\alpha}^r \partial_j) t^r \\
&= \sum_{r=1}^{\infty} \alpha_j T_{-a}^{[1]} \frac{(-1)^r}{(r-1)!} T_{-a-1}^{[r-1]} L_{\alpha}^r t^r + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} L_{\alpha}^r t^r \partial_j \\
&= -\alpha_j T_{-a}^{[1]} u_{a+1} L_{\alpha} t + u_a \partial_j.
\end{aligned}$$

The formula (3.15) holds also, since

$$L_\alpha u_a = L_\alpha \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} L_\alpha^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a-1}^{[i]} L_\alpha^{i+1} t^i = u_{a+1} L_\alpha.$$

For the last equation, we have

$$\begin{aligned} vT_{-a}^{[1]} &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} L_\alpha^i t^i T_{-a}^{[1]} = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} (T - a - i) L_\alpha^i t^i \\ &= T_{-a}^{[1]} v_a - \sum_{i=1}^{\infty} \frac{1}{(i-1)!} (T + a) T_{a-1}^{[i-1]} L_\alpha^i t^i \\ &= T_{-a}^{[1]} v_a - T_a^{[1]} v_{a-1} L_\alpha t. \end{aligned}$$

This completes the proof of the lemma. \square

Proof of Theorem 2.6. For arbitrary elements L_β with $\beta \in \Gamma \setminus \{0\}$, and $\partial_j \in \mathcal{L}(\Gamma)$, $j = 1, 2$, we have

$$\begin{aligned} \Delta(L_\beta) &= \mathcal{F} \Delta_0(L_\beta) \mathcal{F}^{-1} = \mathcal{F} (L_\beta \otimes 1) \mathcal{F}^{-1} + \mathcal{F} (1 \otimes L_\beta) \mathcal{F}^{-1} \\ &= \mathcal{F} (L_\beta \otimes 1) F + \mathcal{F} (1 \otimes L_\beta) F \\ &= \mathcal{F} F_{-b} (L_\beta \otimes 1) + \mathcal{F} \sum_{l=0}^{\infty} (-1)^l F_l (T^{(l)} \otimes L_{\beta+l\alpha} c_l t^l) \\ &= (1 \otimes (1 - L_\alpha t)^b) (L_\beta \otimes 1) \\ &\quad + \sum_{l=0}^{\infty} (-1)^l (1 \otimes (1 - L_\alpha t)^{-l}) (T^{(l)} \otimes L_{\beta+l\alpha} c_l t^l) \\ &= L_\beta \otimes (1 - L_\alpha t)^b + \sum_{l=0}^{\infty} (-1)^l T^{(l)} \otimes (1 - L_\alpha t)^{-l} L_{\beta+l\alpha} c_l t^l, \end{aligned}$$

where $b = a_1 \beta_1 + a_2 \beta_2$, and

$$\begin{aligned} \Delta(\partial_j) &= \mathcal{F} \Delta_0(\partial_j) \mathcal{F}^{-1} = \mathcal{F} (\partial_j \otimes 1 + 1 \otimes \partial_j) F \\ &= \mathcal{F} (\partial_j \otimes 1) F + \mathcal{F} (1 \otimes \partial_j) F \\ &= \mathcal{F} F(\partial_j \otimes 1) + \mathcal{F} (F_1(T^{(1)} \otimes \alpha_j L_\alpha t) + F(1 \otimes \partial_j)) \\ &= \partial_j \otimes 1 + 1 \otimes \partial_j + 1 \otimes (1 - L_\alpha t)^{-1} (T^{(1)} \otimes \alpha_j L_\alpha t) \\ &= \partial_j \otimes 1 + 1 \otimes \partial_j + \alpha_j T^{(1)} \otimes (1 - L_\alpha t)^{-1} L_\alpha t, \quad j = 1, 2, \end{aligned}$$

$$\begin{aligned} S(L_\beta) &= u^{-1} S_0(L_\beta) u = -v L_\beta u = -v u_b \left(\sum_{l=0}^{\infty} c_l L_{\beta+l\alpha} T_1^{(l)} t^l \right) \\ &= -(1 - L_\alpha t)^{-b} \left(\sum_{l=0}^{\infty} c_l L_{\beta+l\alpha} T_1^{(l)} t^l \right), \end{aligned}$$

$$\begin{aligned}
S(\partial_j) &= u^{-1}S_0(\partial)u = -v\partial_j u = -v(-\alpha_j T^{[1]}u_1 L_\alpha t + u\partial_j) \\
&= \alpha_j(Tv - Tv_{-1}L_\alpha t)u_1 L_\alpha t - \partial_j \\
&= \alpha_j Tvu_1 L_\alpha t - \partial_j Tv_{-1}u_2 L_\alpha^2 t^2 - \partial_j \\
&= \alpha_j T(1 - L_\alpha t)^{-1}L_\alpha t - \alpha_j T(1 - L_\alpha t)^{-1}L_\alpha^2 t^2 - \partial_j \\
&= \alpha_j T(1 - L_\alpha t)^{-1}(L_\alpha t - L_\alpha^2 t^2) - \partial_j. \quad \square
\end{aligned}$$

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